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Searching for a unified theory of vortical flows.

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Abstract

Dramatic amplification of rotational motion can occur in vortices of very different scales, which can be observed in a bathtub or the atmosphere. In these vortices, the flow converges towards the axis of rotation and the intensity of rotational motion increases due to conservation of the angular momentum. Destructive nature of hurricanes and tornadoes serves as an illustration of the possible magnitudes of this amplification. We briefly review the theory of compensating regime that can be expected to characterise properties of vortices of different scales and investigate whether available experimental and observational data (both historic and recent) tend to support this theory.

Introduction

Vortical flows have been repeatedly investigated in the literature and many publications are dedicated to this topic. We first mention Theodore Fujita, who in his classical work on vortices in planetary atmospheres [10] introduced a unified treatment of the vortical motion of different scales starting from a lab vortex (that is referred to here as a bathtub vortex) and finishing with the largest vortices in the atmosphere. Among many publications, the works on the strong vortex approximation [5, 21, 23], which introduce asymptotic analysis of vortical flows with dominant influence of vorticity, are relevant to the present analysis. Among conventional vortices, such as vortices introduced by Rott [28], Long [22] and Burgers [3], the scheme of the Burgers vortex, which combines axisymmetric rotation with centripetal flow and axial stretch above the ground, is most related to our consideration. The present work gives a very brief introduction into the theory of compensating regime [14, 15, 16, 17] and considers if this theory is applicable to vortices of different scales and supported by available experimental and observational data.

Compensating regime in vortical flows

Consider incompressible axisymmetric flow with vorticity. The rotational component is characterised by the circulation $\gamma = v_\theta r$ and the axial vorticity ω_z, which are linked by the equation

$$\omega_z = -\frac{1}{r}\frac{\partial \gamma}{\partial r} = -\frac{1}{r}\frac{\partial v_\theta r}{\partial r} \tag{1}$$

The flow image on the r-z plane depends on the value of tangential vorticity

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \tag{2}$$

whose production is controlled by the dimensionless parameter K [14, 15, 16, 17]

$$K^2 = \frac{\gamma \omega_z}{v_*^2} \tag{3}$$

which we call the rotational vorticity parameter. The value v_* represents the characteristic value of translational velocity (with components v_z and v_r) with the axial velocity component v_z being a common choice for v_*. If K is small, no significant tangential vorticity can be present in the flow while large values of K indicate significant production of tangential vorticity. The parameter K is related to most common parameters, which characterise the relative intensity of rotation in the flow, by the equation

$$K^2 = \frac{S}{\mathrm{Ro}} \tag{4}$$

That is the rotational vorticity parameter represents a ratio of the swirl ratio S and the inverse Rossby number Ro, which are defined by

$$S = \frac{\gamma}{v_* r}, \quad \mathrm{Ro} = \frac{v_*}{\omega_z r} \tag{5}$$

Vortical flows with intensive rotation are commonly characterised by existence of an extended region where the flow is directed towards the center and rotation is significantly amplified. This region is limited by the core of the flow at small radii (say $r = r_0$) and by the outer conditions at large radii (say $r = r_1$) while covering the range of radii $r_0 \leq r \leq r_1$ changing at least one order of magnitude $r_1/r_0 \gtrsim 10$. A smaller radius range would not be sufficient for a significant amplification of the rotation. The theory of compensating regime states that the parameter K should remain approximately the same through the amplification region. Indeed, excessively small values of K would make ω_θ negligibly small, which, as shown below, results in increase in K. Excessively large values of K are also not possible since the amount of ω_θ produced in the flow is limited by the flow geometry — very large K are inconsistent with ability of the flow to comply with various boundary conditions and likely to cause bifurcations and/or growth of instabilities The compensating regime is based on assumption $K \sim 1$ where we constrain the order of magnitude of the parameter K rather than its exact value.

For the sake of simplicity, we assume the stream function ψ can be approximated in the region of interest by the power law expression $\psi \sim r^\alpha z$ — this equation is consistent with both the strong vortex approximation and with potential flow. Note that potential flow (i.e. a flow with negligible tangential vorticity ω_θ) corresponds to $\alpha = 2$. The main parameters of the flow are given by the equations

$$\psi = c_0 r^\alpha z, \; v_r = -c_0 r^{\alpha-1}, \; v_z = \alpha c_0 r^{\alpha-2} z,$$
$$\omega_z = \frac{-c_1}{r v_r} = \frac{c_1}{c_0 r^\alpha}, \; \gamma(r) = \gamma_0 + \gamma_1(r) \tag{6}$$

where

$$\gamma_1(r) = \frac{c_1}{c_0(2-\alpha)} r^{2-\alpha}$$

and c_0 and c_1 are constants. The parameter K is given by the equation

$$K^2 \sim \frac{\gamma}{r^{3\alpha-4}} \tag{7}$$

which has two obvious limits that can be characterised by simple power laws

$$K^2 \sim \left\{ \begin{array}{ll} \frac{\gamma_0}{r^{3\alpha-4}} & \gamma_0 \gg \gamma_1 \\ \frac{1}{r^{4\alpha-6}} & \gamma_0 \ll \gamma_1 \end{array} \right\} \tag{8}$$

If K is small then ω_θ is not generated in sufficient quantities to affect the flow. In this case, the flow image on the r-z plane is potential (i.e. $\omega_\theta \approx 0$) and this corresponds to $\alpha = 2$. In this case K increases towards the center until it reaches values sufficiently large to generate tangential vorticity ω_θ and reduce α. The requirement of $K \sim 1$ does not generally result in an exact power law but, according to equation (8), $\alpha = \alpha^*$ should lay in the range

$$\frac{4}{3} \leq \alpha^* \leq \frac{3}{2} \tag{9}$$

where α^* is called the compensating value of the exponent α. As discussed in the previous publications [14, 15, 16, 17], the flow compensates for over- and under-production of tangential vorticity by adjusting α to its compensating value α^*, which is given by equation (9) and ensures that the order of the rotational vorticity parameter K remains consistent with the flow in the amplification region.

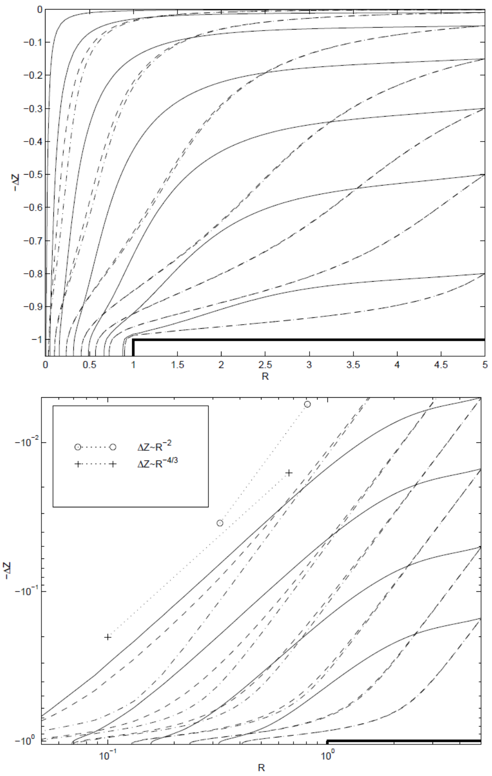

Figure 1: Streamlines in a bathtub flow with progressively strong vorticity obtained in simulations [16]. Coordinates normal (top) and logarithmic (bottom); different line types show simulations with different values of K, : — $K \approx 0.31$, - - - - $K \approx 0.0045$, -.-.- $K = 0$; symbols: + ($\alpha = 4/3$), o ($\alpha = 2$)

Numerical simulations

The 4/3 and 3/2 power laws are devised for effectively inviscid flows and may disappear in the viscous core under dominant influence of viscosity. Detecting this power law in numerical simulations requires obtaining stable solutions under conditions of low viscosity. Klimenko [16] has performed inviscid calculations of an axisymmetric vortical bathtub-like flow and found that the region of the 4/3 power law appears in the flow and grows in size as the parameter K_* increases. Although the obtained numerical solutions of the inviscid vorticity equations are stable, special measures were required to ensure convergence of the solution. The calculations were performed for the case of $\gamma_0 \gg \gamma_1$ and $\alpha^* = 4/3$ was expected and observed in calculations as shown in Figure 1.

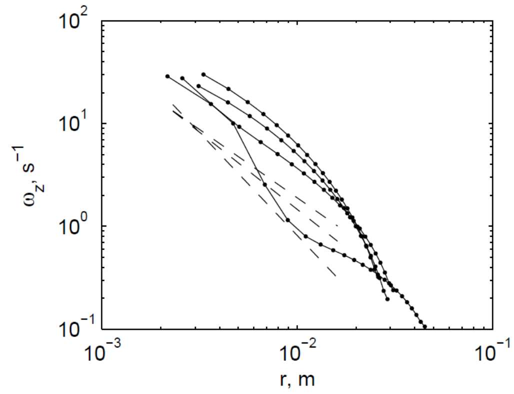

Figure 2: Axial vorticity vs radius in bathtub flow [29]. The solid curves represent ω_z for drains with diameters 20, 30, 40 and 50 mm (from the bottom curve to the top curve). The dashed lines show the exponents of $\alpha = 2$, 3/2 and 4/3 in $\omega_z \sim 1/r^\alpha$.

Bathtub vortex

The power laws can also be detected in experiments on vortical flows in a bathtub. Shiraishi and Sato [29] measured experimental profiles of $v_\theta = \gamma/r$ for vortical flow in a bathtub. As expected v_θ does not have any substantial dependence on z and decreases rapidly with increasing r. The curves $v_\theta(r)$ in Ref.[29] are smooth and allow for numerical differentiation by polynomial approximations of the curves. The axial vorticity profiles are presented in Figure 2. The lines of $\omega_z \sim 1/r^\alpha$ are also shown in the figure for $\alpha = 4/3$, 3/2 and 2. The vorticity profile in the case of the smallest drain hole (20mm) and the weakest rotation in the flow appears to be less regular (the overall slope of the curve in this case seems to be closer to $\alpha = 2$) while the near axis asymptotes of $\omega_z(r)$ for other cases (the drains of 30, 40, and 50 mm) indicate α varying between 3/2 and 4/3.

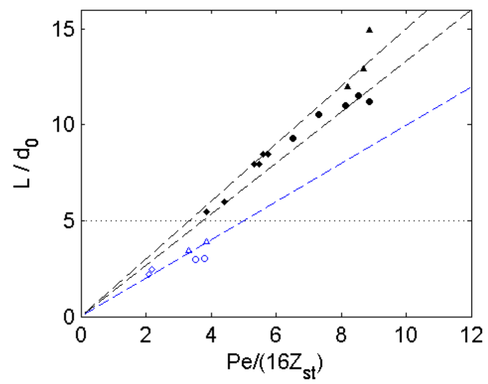

Figure 3: Length versus Pe number. Experiments [4] open symbols (below the dotted line) – no rotation; solid symbols (above the dotted line) – with rotation. Dashed lines – theory [18, 19] $\alpha = 4/3$, 3/2 and 2 (from top to bottom); d_0 is the diameter of the fuel source.

Firewhirls

Firewhirls are fires characterised by the presence of a strong rotation in the flow, high burning rate and elongated flame [30]. Klimenko and Williams [18, 19] have recently extended analysis of Kuwana et. al. [20] and introduced a theory, which determines the flame length and uses velocity approximations based on the compensating regime. While Refs.[18, 19] take into account presence of the viscous core and discusses influence of the density change, Figure 3 presents a simplified treatment linked to the characteristic values of α used in the rest of the paper: 2, 3/2 and 4/3. The value $\alpha = 2$ is associated with irrotational flows, while the compensating values 3/2 and 4/3 are applicable to the case when rotation in the flow is strong. The experimental points of Chuah et al. [4] shown in Figure 3 are in a good agreement with the theoretical prediction.

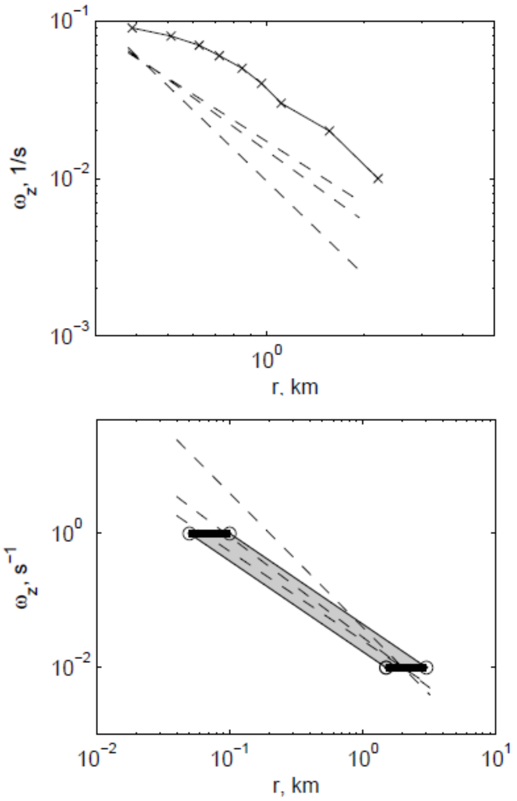

Figure 4: Axial vorticity in tornadoes: top - tornado 4 of the McLean storm [8], bottom - estimates of typical parameters of supercell tornadoes from various sources [17]. The dashed lines show the exponents of $\alpha = 2$, 3/2 and 4/3.

Tornadoes

Although tornadoes generate faster winds than hurricanes, they are more susceptible to atmospheric fluctuations. Even large supercell tornadoes are significantly affected by atmospheric irregularities. There are very few direct measurements of wind profiles in tornadoes and measurements may suffer from under-resolving the core of tornadoes. Wurman and Gill [31] conducted high resolution measurements of a F4 tornado formed in a supercell storm near Dimmitt (Texas) in 1995 and reported $\beta = 0.6 \pm 0.1$ in $v_\theta \sim 1/r^\beta$ (γ_0 is small in the reported profile) that corresponds to $\alpha = \beta + 1 = 1.6 \pm 0.1$. The value of $\alpha^* = 3/2$ is within this range. Figure 4 (bottom) shows the range of typical tornado parameters taken from various sources and summarised in Ref. [17]. These estimates are consistent with the compensating values of the exponent α.

Dowell and Bluestein[8] reported characteristics of several tornadoes that appeared in the 1995 McLean (Texas) storm based on Doppler radar measurements. Among these tornadoes, tornado 4, which reached F4-F5 on the Fujita scale, was the strongest, largest and most stable tornado. Unlike in many other tornadoes disturbed by atmospheric irregularities in surrounding flows, the axial vorticity in tornado 4 was fairly uniform up to AGL (above ground level) of more than 4 km and it persisted for more than an hour. The results are plotted in Figure 4 (top). The circles have been determined from the contour plot of the constant values of vorticity by calculating the average effective radius of each contour line. The error bars show the standard deviations in evaluating these averages. Both values match reasonably well; the increasing difference at $r > 1$km is explained by the difficulty of evaluating $\omega_z(r)$ from $\gamma(r)$ due to an increasingly non-axisymmetric structure of the flow appearing at these radii. The exponents of the compensating regime produce a good match to the measured vorticity within the range of 400m$< r <$ 1.5km.

Hurricanes

Hurricanes, which are also called typhoons or tropical cyclones, are by far the largest and most powerful vortices in atmosphere. Influence of hurricane winds might be detected as far as 1000km from the hurricane eye. The fact that the exponent of $\beta = 0.5$ in $v_\theta = \gamma/r \sim 1/r^\beta$ (which corresponds to $\alpha = \beta + 1 = 3/2$) represents a reasonable empirical approximation for the measurements of the rotational velocities in hurricanes was known for a long time and is mentioned in many publications (see [27, 11, 7]). Riehl [27] noted that assuming both the moment of the tangential component of the surface stress $r\sigma_\theta$ and the drag coefficient C_D to be independent of r is sufficient (but not necessary) for α to be 1.5. Pearce [26] put forward arguments supporting this assumption. The data reported by Hawkins and Rubsam [12] and by Palmen and Riehl [25] indicate, however, that $C_D \sim 1/r^\zeta$ with ζ ranging between 0.4 and 0.7 while Palmen and Riehl [25] determined that, on average, $r\sigma_\theta \sim 1/r^{0.6}$. In his thermodynamic theory of steady tropical cyclones, Emanuel [6] demonstrated that $\alpha \approx 1.5$ just outside the radius of maximal winds is consistent with typical temperature changes on the sea surface and in the tropopause.

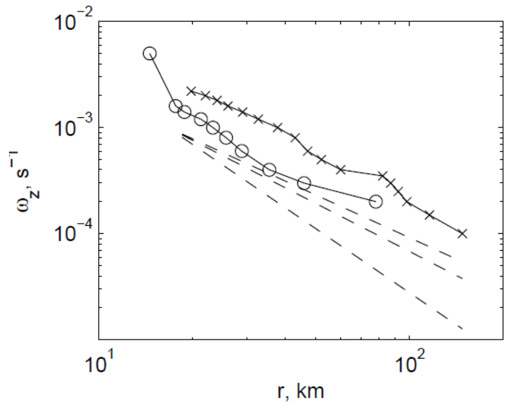

Figure 5: Axial vorticity distributions for hurricanes x – Hilda [12] and o – Inez [13]. The dashed lines show the exponents of $\alpha = 2$, 3/2 and 4/3.

Comparison of the power laws of the compensating regime with observations using axial vorticity is more direct than that using tangential velocity. Hawkins & Rubsam [12] and Hawkins & Imbembo [13] reported radial vorticity distributions and other characteristics for two hurricanes, Hilda (1964) and Inez (1966). Hurricane Inez was a relatively small but very intense hurricane while the parameters of hurricane Hilda were more or less typical for large category 4 cyclones. The vorticity profiles reported for Inez and Hilda also do not show any significant dependence on z at lower altitudes. Axial vorticity profiles in Hilda has an irregularity at the altitudes above several kilometers while ω_z in Inez remains more regular. The dependence of ω_z on r in hurricanes Hilda and Inez is shown in Figure 5. The slope of the curves $\omega_z \sim 1/r^{\alpha}$ exhibits some variations but is generally consistent with the lines of $\alpha = 3/2$ and $\alpha = 4/3$.

The most comprehensive analysis of the exponent in vortical flows by Mallen et. al. [24] reported averages for axisymmetric tangential velocity and axial vorticity distribution in tropical storms involving 251 different cases (while distinguishing pre-hurricanes (<30m/s), minor hurricanes (30-50 m/s) and major hurricanes (>50 m/s) by their maximal tangential winds). The best approximation for the exponent is $\alpha = 1.37$ was determined as the average over all storms with standard deviation of 0.14 while the averages of 1.31 and 1.48 were suggested for the weakest and the strongest storms. These values are quite close to $4/3$ and $3/2$ advocated here.

Experimental [24]			Theoretical [14] – [17]	
hurricane	α	n	Case	α
pre-hurricane	1.31	73	dominant core	$4/3 \approx 1.33$
minor	1.35	106	dominant core	$4/3 \approx 1.33$
major	1.48	72	calm core	$3/2 \approx 1.5$
total average	1.37	251	average	1.38

Table 1. Measured exponent α averaged over n hurricanes versus predictions by the compensating regime theory.

Conclusion

The theory of compensating regime predicts that, for a developed vortex with a strong vorticity, the value of exponent α tend to fall below $\alpha = 2$. If vorticity is sufficiently strong, α is expected to reach the compensating values of the exponent, which lay in the rage $4/3 \leq \alpha \leq 4/3$. Experimental and observational results, taken across a very wide range of phenomena of different scales, tend to support these predictions. We note however, that the exponents are subject to significant variations and noticeable deviations from any fixed value of the exponent — the vortices are strongly affected by many factors including variable atmospheric conditions. There is, however, a case, which allows for a more accurate comparison: average value of the exponents evaluated over many hurricanes indicate a very good quantitative agreement with the theory.

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